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Properties of Slant Toeplitz Operators on the Torus

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ABSTRACT

This paper is an attempt to study various structural and spectral properties of the class of slant Toeplitz operators on the Lebesgue space of the torus. The paper discusses hyponormality and isometric behaviour of these operators along with the study of their adjoint. Spectral radius formula for these operators is derived and it is proved that the spectrum of slant Toeplitz operators induced by invertible symbols contains a closed disk.

Keywords: Toeplitz operator, slant Toeplitz operator, hyponormal operator, weighted composition operator, Laurent operator.

1. Motivation

There are a few classes of operators on Hilbert spaces, other than normal operators and compact operators, which have been studied extensively and possess detailed information. An exceptional class regarding which vast literature is available is the class of Toeplitz operators. The study of these operators was initiated near the beginning of the twentieth century by Toeplitz (1911). Not only the theory of Toeplitz operators is itself interesting, but in addition has connections with C^*-algebras, function theory and many other areas of operator theory.

Let $L^2(= L^2(\mathbb{T}))$ denote the Hilbert space of all complex-valued measurable functions on unit circle $\mathbb{T}$ such that $\int_{\mathbb{T}} |f|^2 d\mu < \infty$, where $d\mu$ is the normalized Lebesgue measure on $\mathbb{T}$. This space has a canonical orthonormal basis $\{e_n\}$ given by $e_n(z) = z^n$, for each $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. The Hardy-Hilbert space $H^2(= H^2(\mathbb{T}))$ is the closed linear span of $\{e_n : n \geq 0\}$ and is a subspace of L^2. Toeplitz operators on H^2 are the compressions of Laurent operators on L^2 and their representing matrices possess a constancy along the diagonals parallel to the main one. Even though the difference in definitions of these two classes of operators may seem trivial, their theories are profoundly different. The spectral, algebraic and C*-algebraic aspects of Toeplitz operators have culminated into a deep and substantial theory. For these operators have such appeal and wide range of applications, many mathematicians over the years have come up with different generalizations of Toeplitz operators (see Power (1980) and the references therein).

The operators whose matrices are derived by eliminating every alternate row from the matrices of Laurent operators constitute the class of slant Toeplitz operators. These were brought into attention by Ho (1996) in 1996. Ho studied spectral as well as structural properties of these operators. More recently, the adjoints of these operators were discussed in details (see Ho (1997, 2001)). The similarity of the adjoint of a slant Toeplitz operator with a constant multiple of shift of the same kind was established under some assumptions on the inducing symbol.

Meanwhile, the study of Toeplitz operators was lifted to the Hardy space of the bidisk by Gu and Zheng (1997). They obtained the condition(s) under which the semi-commutator $T_fT_g - T_{fg}$ on the bidisk is compact. Certain results regarding semi-commutator of Toeplitz operators obtained on bidisk were found to be false on the unit disk. In more recent times, commuting Hankel and Toeplitz operators on the Hardy space of the bidisk are characterized by

Lu and Zhang Lu and Zhang (2010), while commuting Toeplitz operators on the bidisk are described in Ding et al. (2012).

The function theory on bidisk is quite different than on the unit disk and much less understood. However, we put to use the existing literature regarding the multiple Fourier series on the torus $\mathbb{T}^2$ (see Gu and Zheng (1997) and the references therein) and motivated by the work of Ho (2001), study the nature of slant Toeplitz operators on the Lebesgue space of the torus.

In Section 2, we collect some known facts and results. In Section 3, we discuss some elementary properties like norm, isometric behaviour, normality and hyponormality of slant Toeplitz operators on $L^2(\mathbb{T}^2)$. Section 4 investigates the spectral structure of these operators. We conclude our paper with some illustrations based on the findings of the paper.

2. Preliminaries

Let $\mathbb{C}$ denote complex plane. The torus $\mathbb{T}^2$ is the subset of $\mathbb{C}^2$ which is Cartesian product of two copies of $\mathbb{T}$. Let $d\nu$ be the normalized Haar measure on $\mathbb{T}^2$ and $L^2(\mathbb{T}^2)(= L^2(\mathbb{T}^2, d\nu))$ be the usual Lebesgue space of $\mathbb{T}^2$. As in Stein and Weiss (1971), we consider multiple Fourier series on $\mathbb{T}^2$, which can be viewed as the Fourier transformation on $L^1(\mathbb{T}^2)$. The Fourier transformation of $f \in L^1(\mathbb{T}^2)$ on $\mathbb{Z} \times \mathbb{Z}$ is given as $f_{m_1,m_2} = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2}) e^{-i(m_1\theta_1 + m_2\theta_2)} d\theta_1 \, d\theta_2$, where $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$.

If $f_{m_1,m_2} = 0$ for each $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$, then $f \equiv 0$. We also recall that $L^2(\mathbb{T}^2) = \{f = \sum\limits_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} z_1^{m_1} z_2^{m_2} : \|f\|^2 = \sum\limits_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |f_{m_1,m_2}|^2 < \infty\}$ using multiple Fourier series and the set $\{e_{n_1,n_2}(z_1, z_2) = z_1^{n_1} z_2^{n_2}\}_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T}^2)$.

$L^\infty(\mathbb{T}^2)$ denotes the Banach space of all essentially bounded measurable functions on $\mathbb{T}^2$ with $\| \cdot \|_\infty$.

Let M_ϕ denote the Laurent operator on $L^2(\mathbb{T}^2)$ induced by $\phi \in L^\infty(\mathbb{T}^2)$. The notion of a slant Toeplitz operator A_ϕ on $L^2(\mathbb{T}^2)$, induced by the symbol $\phi \in L^\infty(\mathbb{T}^2)$ and defined as $A_\phi f = EM_\phi f$ for each $f \in L^2(\mathbb{T}^2)$, was introduced in Datt and Ohri. Here, the operator E on $L^2(\mathbb{T}^2)$ is defined as $E z_1^{m_1} z_2^{m_2} = z_1^{\frac{m_1}{2}} z_2^{\frac{m_2}{2}}$ if both m_1 and m_2 are even integers and zero otherwise.

If $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$, then for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, the action of A_ϕ and its adjoint A_ϕ^* on the basis elements of $L^2(\mathbb{T}^2)$ is given by $A_\phi(z_1^{n_1} z_2^{n_2}) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-n_1,2m_2-n_2} z_1^{m_1} z_2^{m_2}$ and $A_\phi^*(z_1^{n_1} z_2^{n_2}) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-m_1,2n_2-m_2} z_1^{m_1} z_2^{m_2}$.

The operator equation $M_{z_1 z_2} A = A M_{z_1^2 z_2^2}$ characterizes slant Toeplitz operators on $L^2(\mathbb{T}^2)$. We refer to Datt and Ohri for the structure and some elementary properties of slant Toeplitz operators on $L^2(\mathbb{T}^2)$. The symbols $\sigma_{app}(T), \sigma_p(T), \sigma(T)$ and $\rho(T)$ denote respectively the approximate point spectrum, the point spectrum, the spectrum and the resolvent set of an operator T, while $r(T)$ denotes the spectral radius of T.

3. Basic properties

This section investigates the behaviour of a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ and that of its adjoint. For a symbol $\phi \in L^\infty(\mathbb{T}^2)$ given by $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1}\ z_2^{m_2}$, $\overline{\phi}$ is defined as $\overline{\phi}(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{m_1,m_2} z_1^{-m_1} z_2^{-m_2}$. Clearly $\overline{\phi} \in L^\infty(\mathbb{T}^2)$ and has same norm as that of ϕ. In the initial attempt, we decompose a multiplication operator on $L^2(\mathbb{T}^2)$ into slant Toeplitz operators.

Lemma 3.1. *Let $\phi \in L^\infty(\mathbb{T}^2)$. Then, $\psi = E\overline{\phi} \in L^\infty(\mathbb{T}^2)$ and $EA_\phi^* = M_\psi$.*

Proof. Using the definition of A_ϕ^* and E, we obtain that for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$,

$$\begin{aligned} EA_\phi^*(z_1^{n_1} z_2^{n_2}) &= E\Big(\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-m_1,2n_2-m_2} z_1^{m_1} z_2^{m_2}\Big) \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-2m_1,2n_2-2m_2} z_1^{m_1} z_2^{m_2} \\ &= \Big(\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{-2m_1,-2m_2} z_1^{m_1} z_2^{m_2}\Big).(z_1^{n_1} z_2^{n_2}) \\ &= \psi(z_1, z_2).(z_1^{n_1} z_2^{n_2}), \end{aligned}$$

where $\psi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{-2m_1,-2m_2} z_1^{m_1} z_2^{m_2} = E\overline{\phi}(z_1, z_2) \in L^2(\mathbb{T}^2)$. Using the linearity of the operator EA_ϕ^*, we find that $EA_\phi^* f = \psi.f$ for each

$f \in L^2(\mathbb{T}^2)$. Now, utilizing Halmos (1982), we obtain that $\psi \in L^\infty(\mathbb{T}^2)$. Therefore, $EA_\phi^* = M_\psi$, where $\psi = E\overline{\phi}$. This completes the proof. □

Next, we compute the product of a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ and its adjoint. We find that the product is a Laurent operator as is justified in the following result.

Proposition 3.1. *For $\phi, \psi \in L^\infty(\mathbb{T}^2)$, we have the following.*

(1) $M_\phi^* = M_{\overline{\phi}}$.

(2) $M_\phi M_\psi = M_{\phi\psi}$

(3) $A_\phi A_\phi^* = M_{E|\phi|^2}$.

Proof. (1) and (2) follow immediately using the definition of a Laurent operator. We prove only (3). Using the definition of a slant Toeplitz operator and Lemma 3.1, we obtain $A_\phi {A_\phi}^* = (EM_\phi)(EM_\phi)^* = EM_{|\phi|^2}E^* = E(EM_{|\phi|^2})^* = EA_{|\phi|^2}{}^* = M_{E|\phi|^2}$. □

The above proposition leads us to the norm of a slant Toeplitz operator on $L^2(\mathbb{T}^2)$.

Theorem 3.1. *For $\phi \in L^\infty(\mathbb{T}^2)$, then $\|A_\phi\| = \sqrt{\|E|\phi|^2\|_\infty}$.*

With the help of Theorem 3.1, we obtain that a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ can never be an isometry.

Theorem 3.2. *No slant Toeplitz operator on $L^2(\mathbb{T}^2)$ is isometric.*

Proof. Let A_ϕ on $L^2(\mathbb{T}^2)$ be an isometry, where $\phi \in L^\infty(\mathbb{T}^2)$ is given as $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$. Then, $\|A_\phi 1\| = \|A_\phi z_1\| = \|A_\phi z_2\| = \|A_\phi z_1 z_2\| = 1$ and $\|A_\phi \overline{\phi}\| = \|E|\phi|^2\| = \|\phi\|$. This yields that for each $i, j \in \{0, 1\}$,

$$\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\phi_{2m_1-i,2m_2-j}|^2 = 1.$$

As a consequence,

$$\|E|\phi|^2\|^2 = \|\phi\|^2 = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\phi_{m_1,m_2}|^2 = \sum |\phi_{2m_1,2m_2}|^2 + \sum |\phi_{2m_1-1,2m_2}|^2$$
$$+ \sum |\phi_{2m_1,2m_2-1}|^2 + \sum |\phi_{2m_1-1,2m_2-1}|^2 = 4.$$

However, Theorem 3.1 provides $\|A_\phi\| = \sqrt{\|E|\phi|^2\|_\infty}$. Also, $\|A_\phi\| = 1$ since A_ϕ is assumed to be an isometry. Putting together all this information, we arrive at the following.

$$\|E|\phi|^2\| = 2 \text{ and } \|E|\phi|^2\|_\infty = 1,$$

which is a contradiction since $\|\psi\| \leq \|\psi\|_\infty$ for any $\psi \in L^\infty(\mathbb{T}^2)$. This completes the proof. □

We now proceed to investigate the existence of a hyponormal slant Toeplitz operator in our next theorem.

Theorem 3.3. *The only hyponormal slant Toeplitz operator on $L^2(\mathbb{T}^2)$ is the zero operator.*

Proof. Suppose that A_ϕ, where $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$, be a hyponormal operator. This implies that for each $f \in L^2(\mathbb{T}^2)$,

$$\|A_\phi f\| \geq \|A_\phi^* f\|. \tag{1}$$

In particular, for $f(z_1, z_2) = 1$, inequality (1) provides that $\|A_\phi 1\|^2 \geq \|A_\phi^* 1\|^2$. Hence, $\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\phi_{2m_1,2m_2}|^2 \geq \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\overline{\phi}_{-m_1,-m_2}|^2$. Consequently,

$$\phi_{2m_1-i,2m_2-j} = 0, \tag{2}$$

for all $(m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}$ and for $(0, 0) \neq (i, j) \in \{0, 1\} \times \{0, 1\}$.

Again, substituting $f(z_1, z_2) = z_1 z_2$ in (1) and using the structure of A_ϕ, we get that $\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\phi_{2m_1-1,2m_2-1}|^2 \geq \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |\overline{\phi}_{2-m_1,2-m_2}|^2$.

This, together with the condition (2) helps us to obtain that $\phi_{2m_1,2m_2} = 0$ for all $(m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}$. Therefore, $\phi = 0$ and hence $A_\phi = 0$. □

Since every normal operator is hyponormal, so the following observation is immediate.

Corollary 3.1. *A slant Toeplitz operator on $L^2(\mathbb{T}^2)$ is normal if and only if $\phi = 0$.*

It is easy to observe that every isometry is a hyponormal operator, for if A is an isometry, then $\|A^*f\| \leq \|A^*\| \, \|f\| = \|f\| = \|Af\|$, for each f in domain of A. So, a quick observation from Theorem 3.3 is that a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ can't be an isometry. This was proved independently in Theorem 3.2 as well.

The fact that A_ϕ is non-isometric brings us to our next question. We try to determine whether some A_ϕ is a partial isometry or a co-isometry. A positive answer to this question can be seen with the help of the operator $A_1(= E)$ which satisfies $E = EE^*E$ and $EE^* = I$.

In the results that follow, we obtain the condition(s) on the inducing symbol ϕ so that A_ϕ is a partial isometry or a co-isometry.

Theorem 3.4. *A slant Toeplitz operator A_ϕ on $L^2(\mathbb{T}^2)$ is a partial isometry if and only if $\phi = \phi E^*E|\phi|^2$. Particularly, if ϕ is invertible, then A_ϕ is a partial isometry if and only if $E^*E|\phi|^2 = 1$.*

Proof. A_ϕ is a partial isometry if and only if $A_\phi = A_\phi A_\phi^* A_\phi = M_{E|\phi|^2} A_\phi = A_{\phi E^*E|\phi|^2}$. Now, due to the injectivity of the map $\phi \to A_\phi$ in (Datt and Ohri), the result is immediate. □

Theorem 3.5. *A necessary and sufficient condition for a slant Toeplitz operator A_ϕ on $L^2(\mathbb{T}^2)$ to be a co-isometry is that $|\phi(\frac{\theta_1}{2}, \frac{\theta_2}{2})|^2 + |\phi(\frac{\theta_1+2\pi}{2}, \frac{\theta_2)}{2}|^2 + |\phi(\frac{\theta_1}{2}, \frac{\theta_2+2\pi}{2})|^2 + |\phi(\frac{\theta_1+2\pi}{2}, \frac{\theta_2+2\pi}{2})|^2 = 4$ for almost everywhere (a.e.) $\theta_1, \theta_2 \in [0, 2\pi)$.*

Proof. We arrive at the result using simple computations. For, consider any $f \in L^2(\mathbb{T}^2)$, then we have

$$\begin{aligned}\|A_\phi^* f\|^2 &= \|M_{\overline{\phi}} E^* f\|^2 \\ &= (\frac{1}{2\pi})^2 \int_0^{2\pi}\int_0^{2\pi} |\phi(\theta_1,\theta_2)|^2\ |f(2\theta_1,2\theta_2)|^2 d\theta_1 d\theta_2 \\ &= (\frac{1}{2\pi})^2 \int_0^{4\pi}\int_0^{4\pi} |\phi(\frac{\theta_1}{2},\frac{\theta_2}{2})|^2\ |f(\theta_1,\theta_2)|^2 \frac{d\theta_1}{2}\frac{d\theta_2}{2} \\ &= (\frac{1}{2\pi})^2(\frac{1}{4}) \int_0^{2\pi}\int_0^{2\pi} \Big[\Big|\phi(\frac{\theta_1}{2},\frac{\theta_2}{2})\Big|^2 + \Big|\phi(\frac{\theta_1+2\pi}{2},\frac{\theta_2)}{2}\Big|^2 \\ &\quad + \Big|\phi(\frac{\theta_1}{2},\frac{\theta_2+2\pi}{2})\Big|^2 + \Big|\phi(\frac{\theta_1+2\pi}{2},\frac{\theta_2+2\pi}{2})\Big|^2\Big] \\ &\quad |f(\theta_1,\theta_2)|^2\ d\theta_1\ d\theta_2 \\ &= \|M_\psi f\|^2,\end{aligned}$$

with $\psi(\theta_1,\theta_2) = \sqrt{\frac{\left|\phi(\frac{\theta_1}{2},\frac{\theta_2}{2})\right|^2 + \left|\phi(\frac{\theta_1+2\pi}{2},\frac{\theta_2}{2})\right|^2 + \left|\phi(\frac{\theta_1}{2},\frac{\theta_2+2\pi}{2})\right|^2 + \left|\phi(\frac{\theta_1+2\pi}{2},\frac{\theta_2+2\pi}{2})\right|^2}{4}}$.
The result is now immediate since $\|M_\psi f\| = \|f\|$ if and and only if $|\psi| = 1$ almost everywhere on $\mathbb{T}^2$. □

The above theorem can also be restated in the following manner.

Theorem 3.6. *A_ϕ on $L^2(\mathbb{T}^2)$ is a co-isometry if and only if $|\phi(z_1,z_2)|^2 + |\phi(z_1,-z_2)|^2 + |\phi(-z_1,z_2)|^2 + |\phi(-z_1,-z_2)|^2 = 4$ for a.e. $z_1,z_2 \in \mathbb{T}$.*

Using the above theorem, the following is obtained without any extra efforts.

Corollary 3.2. *If $\phi \in L^\infty(\mathbb{T}^2)$ is unimodular, then A_ϕ is always a co-isometry.*

With our next theorem, we establish a connection between a slant Toeplitz operator and a composition operator on $L^2(\mathbb{T}^2)$. If we consider the mapping $H : \mathbb{T}^2 \to \mathbb{T}^2$ given by $H(z_1,z_2) = (z_1^2, z_2^2)$, then H induces the composition operator (see Singh and Manhas (1993)) C_H on $L^2(\mathbb{T}^2)$ given by $C_H f = f \circ H$ for each $f \in L^2(\mathbb{T}^2)$. Further, if $\psi \in L^\infty(\mathbb{T}^2)$, then the weighted composition operator $C_{\psi,H}$ $(= M_{\overline{\psi}} C_H)$ is a bounded operator on $L^2(\mathbb{T}^2)$, where $C_{\psi,H} f = \overline{\psi}.(f \circ H)$ for each $f \in L^2(\mathbb{T}^2)$. It is interesting to obtain that the adjoint of a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ is a weighted composition operator on $L^2(\mathbb{T}^2)$.

Theorem 3.7. *For $\phi \in L^\infty(\mathbb{T}^2)$, $A_\phi^* = C_{\phi,H}$.*

Proof. Let $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Consider $e_{n_1,n_2}(z_1, z_2) = z_1^{n_1} z_2^{n_2} \in L^2(\mathbb{T}^2)$. Then by definition of A_ϕ^*, we have

$$\begin{aligned} A_\phi^* e_{n_1,n_2}(z_1, z_2) &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-m_1,2n_2-m_2} z_1^{m_1} z_2^{m_2} \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{m_1,m_2} z_1^{-m_1+2n_1} z_2^{-m_2+2n_2} \\ &= \big(\overline{\phi}(z_1, z_2)\big).(z_1^{2n_1} z_2^{2n_2}) \\ &= \big(\overline{\phi}(z_1, z_2)\big).\big(e_{n_1,n_2} \circ H(z_1, z_2)\big). \end{aligned}$$

Since the above relation holds true for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ and the set $\{e_{n_1,n_2} : (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{T}^2)$, the linearity of the operator A_ϕ^* helps to obtain that $A_\phi^* f = \overline{\phi}.(f \circ H)$, for each $f \in L^2(\mathbb{T}^2)$. This provides the desired result. □

In the results that follow, we discuss the structure of the C^*-algebra generated by the set of all slant Toeplitz operators on $L^2(\mathbb{T}^2)$. Lemma 3.1 provides that any Laurent operator M_ϕ can be written as product of $A_1 (= E)$ and A_ψ^*, for $\overline{\psi} = E^*\phi$. Therefore, if $\mathfrak{T}$ and $\mathfrak{M}$ denote respectively the C^*-algebras generated by all slant Toeplitz operators on $L^2(\mathbb{T}^2)$ and all Laurent operators on $L^2(\mathbb{T}^2)$, then $\mathfrak{M} \subseteq \mathfrak{T}$.

This observation, together with the fact that for $\phi \in L^\infty(\mathbb{T}^2)$, $EM_\phi = M_\phi E$ if and only if ϕ is constant (Datt and Ohri), helps us to obtain the following lemma.

Lemma 3.2. *$\mathfrak{T}'' = \mathfrak{B}(L^2(\mathbb{T}^2))$, where $\mathfrak{T}''$ denotes the double commutant of $\mathfrak{T}$ and $\mathfrak{B}(L^2(\mathbb{T}^2))$ denotes the set of all bounded linear operators on $L^2(\mathbb{T}^2)$.*

Proof. Since $\mathfrak{M} \subseteq \mathfrak{T}$, therefore $\mathfrak{T}' \subseteq \mathfrak{M}' = \mathfrak{M}$, where $\mathfrak{T}'$ and $\mathfrak{M}'$ denote respectively the commutants of the sets $\mathfrak{T}$ and $\mathfrak{M}$. Let $T \in \mathfrak{T}'$. Then, $T = M_\psi$ for some $\psi \in L^\infty(\mathbb{T}^2)$. Also since $E = A_1 \in \mathfrak{T}$, so the C^*-algebra generated by E, $(E) \subseteq \mathfrak{T}$ and thus $\mathfrak{T}' \subseteq (E)'$. Hence, we obtain that $M_\psi E = EM_\psi$ and therefore that ψ is constant and T is a constant multiple of the identity operator I. Thus $(I) \subseteq \mathfrak{T}' \subseteq (I)$, where (I) denotes the C^*-algebra generated by I. That is, $\mathfrak{T}' = (I)$ and $\mathfrak{T}'' = \mathfrak{B}(L^2(\mathbb{T}^2))$. □

Using Lemma 3.2 and Von-Neumann double commutant theorem, we arrive at the following.

Theorem 3.8. $\mathfrak{T} = \mathfrak{B}(L^2(\mathbb{T}^2))$, *that is the* C^**-algebra generated by all slant Toeplitz operators on* $L^2(\mathbb{T}^2)$ *is the set of all bounded linear operators on* $L^2(\mathbb{T}^2)$.

4. Spectral properties

This section is aimed at the study of some spectral properties of slant Toeplitz operators on $L^2(\mathbb{T}^2)$. We follow the methods and techniques of Ho (1996) and provide only the outlines of the proofs. Let us begin with the calculation of the spectral radius of a slant Toeplitz operator A_ϕ on $L^2(\mathbb{T}^2)$.

Theorem 4.1. $r(A_\phi) = \lim_{n\to\infty} \|\psi_n\|_\infty^{\frac{1}{2n}}$, *where* $\psi_n = A^n_{|\phi|^2}(1)$.

Proof. Using the principle of mathematical induction, it is easy to see that $A^n_\phi A^{*n}_\phi = M_{\psi_n}$ for each positive integer n. In fact, for $n = 1$, $A_\phi A^*_\phi = M_{E|\phi|^2} = M_{\psi_1}$ follows directly from Proposition 3.1 (1). Further, if we assume that this equation holds for integer k (≥ 1), then making use of the fact that $\psi_k.(E|\phi|^2) = \psi_{k+1}$, we arrive at our claim. Finally, using Gelfand formula for spectral radius, $r(A_\phi) = \lim_{n\to\infty} \|A^n_\phi\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\psi_n\|_\infty^{\frac{1}{2n}}$. □

Corollary 4.1. *The spectral radius of* A_ϕ *induced by* ϕ*, with* ϕ *being an inner function, is 1.*

Lemma 4.1. *If* $\phi \in L^\infty(\mathbb{T}^2)$ *is such that* $\phi^{-1} \in L^\infty(\mathbb{T}^2)$*, then* $\sigma_p(A_\phi) = \sigma_p(A_{\phi(z_1^2,z_2^2)})$.

Proof. Let $\alpha \in \sigma_p(A_\phi)$. Then, there exists a non-zero $f \in L^2(\mathbb{T}^2)$ such that $A_\phi f = \alpha f$. Let $F = \phi f$. Then, since $\phi \neq 0$ a.e., $F \neq 0$ and simple computations provide that $(A_{\phi(z_1^2,z_2^2)})F = \alpha F$. Therefore, $\sigma_p(A_\phi) \subseteq \sigma_p(A_{\phi(z_1^2,z_2^2)})$. Converse follows on similar lines. □

The above lemma helps us to establish the following.

Theorem 4.2. *For any* $\phi \in L^\infty(\mathbb{T}^2)$, $\sigma(A_\phi) = \sigma(A_{\phi(z_1^2,z_2^2)})$.

Proof. Firstly, we prove that $\sigma(A_\phi) \cup \{0\} = \sigma(A_{\phi(z_1^2,z_2^2)}) \cup \{0\}$. Using the definition of A^*_ϕ, we have $\sigma(A^*_\phi) \cup \{0\} = \sigma(M_{\overline{\phi}} E^*) \cup \{0\} = \sigma(E^* M_{\overline{\phi}}) \cup \{0\} = \sigma(A^*_{\phi(z_1^2,z_2^2)}) \cup \{0\}$. Therefore, $\sigma(A_\phi) \cup \{0\} = \overline{\sigma(A^*_\phi)} \cup \{0\} = \overline{\sigma(A^*_{\phi(z_1^2,z_2^2)})} \cup \{0\} = \sigma(A_{\phi(z_1^2,z_2^2)}) \cup \{0\}$.

Next, we observe that 0 always belongs to the point spectrum of $(A_{\phi(z_1^2,z_2^2)})$. Now, if ϕ is invertible, then by Lemma 4.1, we obtain that $0 \in \sigma_p(A_\phi)$. In case, ϕ is not invertible, we obtain that $0 \in \sigma_{app}(A_\phi) \subseteq \sigma_p(A_\phi)$. Therefore, in either case $0 \in \sigma(A_\phi)$. This completes the proof. □

Theorem 4.3. *Let $\phi \in L^\infty(\mathbb{T}^2)$ be invertible. Then, $\sigma(A_\phi)$ contains a closed disk.*

Proof. Let $0 \neq \alpha \in \mathbb{C}$ and P_{ee} be the projection of $L^2(\mathbb{T}^2)$ onto the closed subspace generated by $\{z_1^{2m_1} z_2^{2m_2} : m_1 \text{ and } m_2 \text{ are integers}\}$.

Let $(A^*_{\overline{\phi}^{-1}(z_1^2,z_2^2)} - \alpha I)$ be onto. Choose $0 \neq f_0 \in (I - P_{ee})L^2(\mathbb{T}^2)$. Then, there exists $0 \neq f \in L^2(\mathbb{T}^2)$ such that $f_0 = (A^*_{\overline{\phi}^{-1}(z_1^2,z_2^2)} - \alpha I)f = \alpha E^* M_{\phi^{-1}}(\alpha^{-1} - M_\phi E)f \oplus (-\alpha(I - P_{ee})f)$. However, since $f_0 \in (I - P_{ee})L^2(\mathbb{T}^2)$, E^* is an isometry and $M_{\phi^{-1}}$ is invertible, we obtain that $(\alpha^{-1} - M_\phi E)f = (\alpha^{-1} - A_{\phi(z_1^2,z_2^2)})f = 0$ and thus $\alpha^{-1} \in \sigma_p(A_{\phi(z_1^2,z_2^2)})$. Now, for each $\alpha \in \rho(A^*_{\overline{\phi}^{-1}(z_1^2,z_2^2)})$, $(A^*_{\overline{\phi}^{-1}(z_1^2,z_2^2)} - \alpha I)$ is invertible and thus onto. Therefore, we conclude that $\{\alpha^{-1} | \alpha \in \rho(A^*_{\overline{\phi}^{-1}(z_1^2,z_2^2)})\} \subseteq \sigma_p(A_{\phi(z_1^2,z_2^2)}) = \sigma_p(A_\phi) \subseteq \sigma(A_\phi)$. The compactness of spectrum helps to yield the desired result. □

Some immediate consequences of Theorem 4.3 are the ones listed below.

Corollary 4.2. *The radius of the closed disk contained in the spectrum of A_ϕ is $\frac{1}{r(A_{\overline{\phi}^{-1}})}$.*

Corollary 4.3. $\frac{1}{r(A_{\overline{\phi}^{-1}})} \leq r(A_\phi)$.

Corollary 4.4. *For a unimodular ϕ, $\sigma(A_\phi)$ is the closed unit disk.*

Examples: The following are some examples based on the results obtained in the paper:

(a) Let α be a non-zero complex number. Then, the slant Toeplitz operator A_ϕ, where $\phi(z_1, z_2) = \frac{1}{\sqrt{(1+|\alpha|^2)}}(z_1 z_2 + \alpha)$, is a co-isometry. For, Proposition 3.1 (3) provides that $A_\phi A_\phi^* = M_{E|\phi|^2} = M_1 = I$.

(b) Let $\phi(z_1, z_2) = \frac{1}{2} z_1 z_2 + \frac{\sqrt{3}}{2} z_1^2 z_2^2 \in L^\infty(\mathbb{T}^2)$. Then, $\phi E^* E|\phi|^2 = \phi$ and hence using Theorem 3.4, A_ϕ is a partial isometry. Indeed, $A_\phi A_\phi^* A_\phi = M_{E|\phi|^2} A_\phi = M_1 A_\phi = A_\phi$ and hence A_ϕ is a partial isometry.

(c) Let $\alpha, \beta \in \mathbb{C}$. Then, the operator $\frac{1}{|\alpha|^2+|\beta|^2} A^*_{\alpha z_1 z_2+\beta}$ is an isometry.

(d) Let $\phi(z_1, z_2) = z_1^2 z_2^2 + 1 \in L^\infty(\mathbb{T}^2)$. Then, $\phi\overline{\phi} = |\phi|^2 = 2 + z_1^2 z_2^2 + \overline{z_1}^2 \overline{z_2}^2$ and hence $\psi_1 = E(|\phi|^2) = 2 + z_1 z_2 + \overline{z_1}\ \overline{z_2}$. Similarly, $\psi_2 = E(E(|\phi|^2)|\phi|^2) = 2(2 + z_1 z_2 + \overline{z_1}\ \overline{z_2})$. Moving on in a similar fashion, we obtain that $\psi_n = E(E(\cdots(E(|\phi|^2)|\phi|^2)\cdots|\phi|^2)) = 2^{n-1}(2+z_1 z_2+\overline{z_1}\ \overline{z_2})$. This provides that $\|\psi_n\|_\infty = 2^{n+1}$ and thus Theorem 4.1 provides that $r(A_\phi) = \lim_{n\to\infty} 2^{\frac{n+1}{2n}} = \sqrt{2}$. Also, using Theorem 3.1, we have $\|A_\phi\| = \sqrt{\|E|\phi|^2\|_\infty} = \sqrt{4} = 2$. Therefore, $r(A_\phi) \neq \|A_\phi\|$.

(e) Let $\phi(z_1, z_2) = z_1 z_2 + \alpha$. Then, $r(A_\phi) = \sqrt{1 + |\alpha|^2}$.

5. Conclusion

In our pursuit to lift the study of slant Toeplitz operators defined on $L^2(\mathbb{T})$ to higher dimensional spaces, we describe the structure of these operators on $L^2(\mathbb{T}^2)$. We obtain that a slant Toeplitz operators on $L^2(\mathbb{T}^2)$ can not be an isometry, while the only hyponormal slant Toeplitz operator is the zero operator. Symbols inducing co-isometric and partial isometric operators are identified and a connecting bridge is established between the classes of slant Toeplitz operators and weighted composition operators on $L^2(\mathbb{T}^2)$. Certain spectral properties of these operators are also discussed, in addition to obtaining that the C^*-algebra generated by all slant Toeplitz operators is the set of all bounded linear operators on $L^2(\mathbb{T}^2)$.

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